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SCHRÖDINGER UNCERTAINTY RELATION FOR THE MONOTONE TRIPLE SKEW INFORMATION

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Abstract. We get a Schrödinger uncertainty relation for the monotone triple skew information which was introduced by Yanagi and Kajihara. This information extends the monotone pair skew information as well as Wigner-Yanase-Dyson skew information.

1. Introduction

The skew information plays important roles in quantum mechanics and quantum information theory. The information is in particular a central tool to understand the uncertainty relation. For a density matrix ρ and an observable A, Wigner-Yanase skew information

$$I_\rho(A) := \frac{1}{2}\mathrm{Tr}\big((i[\rho^{1/2}, A])^2\big) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^{1/2} A \rho^{1/2} A) \tag{1.1}$$

was defined in [7]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state (density matrix) ρ and an observable (self-adjoint operator) A. Here the commutator is defined by $[A, B] = AB - BA$.

This quantity was generalized by many authors. First of all the Wigner-Yanase-Dyson skew information has been introduced as [6]

$$I_{\rho,\alpha}(A) := \frac{1}{2}\mathrm{Tr}\Big((i[\rho^\alpha, A])(i[\rho^{1-\alpha}, A])\Big), \quad 0 < \alpha < 1.$$

In [2], Furuichi further generalized this as follows: for a monotone pair (f, g) of operator monotone functions, he defined the (f, g)-skew information by

$$\begin{aligned} I_{\rho,(f,g)}(A) &:= \frac{1}{2}\mathrm{Tr}\Big((i[f(\rho), A])(i[g(\rho), A])\Big) \\ &= \mathrm{Tr}(f(\rho)g(\rho)A^2) - \mathrm{Tr}(f(\rho)Ag(\rho)A). \end{aligned} \tag{1.2}$$

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For $f(x) = x^\alpha$ and $g(x) = x^{1-\alpha}$ $(0 < \alpha < 1)$, $I_{\rho,(f,g)}(A)$ reduces to Wigner-Yanase-Dyson skew information [3, 8, 9]. Recently, Yanagi and Kajihara introduced further generalized skew information by using a triple of operator monotone functions. Given a triple of operator monotone functions f, g, and h, the skew information $I_{\rho,(f,g,h)}(A)$ is defined as follows [10]:

$$I_{\rho,(f,g,h)}(A) := \frac{1}{2}\mathrm{Tr}\Big((i[f(\rho), A_0])(i[g(\rho), A_0])h(\rho)\Big). \tag{1.3}$$

Obviously this generalizes the (f, g)-skew information.

As mentioned above the skew information is a crucial tool to understand the uncertainty relation. There are two types of uncertainty relations. One is Heisenberg-type and the other is Schrödinger type. For the details we refer to [5]. For the monotone pair (f, g)-skew information $I_{\rho,(f,g)}(A)$, the present authors obtained the Heisenberg-type uncertainty relation in [4], which generalized the uncertainty relation associated with Wigner-Yanase-Dyson skew information shown by Yanagi in [9]. Recently this uncertainty relation was generalized by Yanagi and Kajihara via the monotone triple skew information [10] (see (2.12) and (2.13)). In [5], we obtained Schrödinger-type uncertainty relation for the monotone pair skew information. The aim of this paper is to get Schrödinger-type uncertainty relation for the monotone triple skew information.

This paper is organized as follows: In Section 2, we introduce the monotone triple skew information and state the main result (Theorem 2.7). Then we note that it extends the previous results. Section 3 is devoted to the proof of Theorem 2.7.

2. Monotone triple skew information and main result

In this section we introduce the monotone triple skew information and state our main result.

Let M_n (resp. $M_{n,sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). Let D_n be the set of strictly positive elements of M_n and $D_n^1 \subset D_n$ the set of strictly positive density matrices, that is,

$$D_n^1 = \{\rho \in M_n \,;\, \mathrm{Tr}(\rho) = 1,\, \rho > 0\}.$$

Let $\rho \in D_n^1$ be a fixed density matrix. For any $A \in M_{n,sa}$, define $A_0 := A - \mathrm{Tr}(\rho A)I$, where $I \in M_n$ is the identity matrix. The variance $V_\rho(A)$ for ρ and A is defined by $V_\rho(A) = \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}(\rho A))^2 = \mathrm{Tr}(\rho A_0^2)$. The following was introduced in [4].

Definition 2.1. Let $f(x)$ and $g(x)$ be nonnegative continuous operator monotone functions defined on the interval $[0, 1]$. We call the pair (f, g) a compatible in log-increase, monotone pair (CLI monotone pair, in short) if

(a) $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in [0, 1]$,

(b) $f(x)$ and $g(x)$ are differentiable on $(0,1)$ and

$$0 \leq \inf_{0<x<1} \frac{G'(x)}{F'(x)} \leq \sup_{0<x<1} \frac{G'(x)}{F'(x)} < \infty,$$

where $F(x) = \log f(x)$ and $G(x) = \log g(x)$.

Definition 2.2. Let $f(x)$ and $g(x)$ be nonnegative continuous operator monotone functions defined on the interval $[0,1]$. We call the pair (f,g) a compatible in log-increase, anti-monotone pair (CLI anti-monotone pair, in short) if

(a) $(f(x)-f(y))(g(x)-g(y)) \leq 0$ for all $x,y \in [0,1]$,

(b) $f(x)$ and $g(x)$ are differentiable on $(0,1)$ and

$$-\infty < \inf_{0<x<1} \frac{G'(x)}{F'(x)} \leq \sup_{0<x<1} \frac{G'(x)}{F'(x)} \leq 0,$$

where $F(x) = \log f(x)$ and $G(x) = \log g(x)$.

Now we introduce the monotone triple skew information [10]. We fix $\rho \in D_n^1$ and let $f(x)$, $g(x)$, and $h(x)$ be nonnegative continuous operator monotone functions. We assume that (f,g) and (f,h) are CLI monotone pairs (in the case $h=1$ we only assume that (f,g) is a CLI monotone pair).

Definition 2.3. For $A \in M_{n,sa}$, we define

$$\begin{aligned} I_{\rho,(f,g,h)}(A) &:= \frac{1}{2}\mathrm{Tr}\Big((i[f(\rho),A_0])(i[g(\rho),A_0])h(\rho)\Big) \\ &= \frac{1}{2}\Big[\mathrm{Tr}\big(f(\rho)g(\rho)h(\rho)A_0^2\big) + \mathrm{Tr}\big(f(\rho)g(\rho)A_0h(\rho)A_0\big)\Big] \qquad (2.1) \\ &\quad - \frac{1}{2}\Big[\mathrm{Tr}\big(f(\rho)A_0g(\rho)h(\rho)A_0\big) + \mathrm{Tr}\big(g(\rho)A_0f(\rho)h(\rho)A_0\big)\Big], \end{aligned}$$

$$\begin{aligned} J_{\rho,(f,g,h)}(A) &:= \frac{1}{2}\mathrm{Tr}\Big(\{f(\rho),A_0\}\{g(\rho),A_0\}h(\rho)\Big) \\ &= \frac{1}{2}\Big[\mathrm{Tr}\big(f(\rho)g(\rho)h(\rho)A_0^2\big) + \mathrm{Tr}\big(f(\rho)g(\rho)A_0h(\rho)A_0\big)\Big] \qquad (2.2) \\ &\quad + \frac{1}{2}\Big[\mathrm{Tr}\big(f(\rho)A_0g(\rho)h(\rho)A_0\big) + \mathrm{Tr}\big(g(\rho)A_0f(\rho)h(\rho)A_0\big)\Big] \end{aligned}$$

and

$$U_{\rho,(f,g,h)}(A) := \sqrt{I_{\rho,(f,g,h)}(A)J_{\rho,(f,g,h)}(A)}. \qquad (2.3)$$

We call $I_{\rho,(f,g,h)}(A)$ the monotone triple skew information. In order to discuss the Schrödinger-type uncertainty relation we define the (f,g,h)-correlation measure $\mathrm{Corr}_{\rho,(f,g,h)}(A,B)$:

$$\begin{aligned} \mathrm{Corr}_{\rho,(f,g,h)}(A,B) &:= \frac{1}{2}\mathrm{Tr}\Big((i[f(\rho),A_0])(i[g(\rho),B_0])h(\rho)\Big) \\ &= \frac{1}{2}\Big[\mathrm{Tr}\big(f(\rho)g(\rho)h(\rho)A_0B_0\big) + \mathrm{Tr}\big(A_0f(\rho)g(\rho)B_0h(\rho)\big)\Big] \qquad (2.4) \end{aligned}$$

$$-\frac{1}{2}\Big[\mathrm{Tr}\big(A_0f(\rho)B_0g(\rho)h(\rho)\big)+\mathrm{Tr}\big(A_0g(\rho)B_0f(\rho)h(\rho)\big)\Big]$$

for any $A, B \in M_{n,sa}$. When $h = 1$, the monotone triple skew information $I_{\rho,(f,g,h)}(A)$ reduces to the monotone pair skew information $I_{\rho,(f,g)}(A)$ and we write

$$\begin{aligned}
I_{\rho,(f,g)}(A) &:= I_{\rho,(f,g,1)}(A),\\
J_{\rho,(f,g)}(A) &:= J_{\rho,(f,g,1)}(A),\\
U_{\rho,(f,g)}(A) &:= U_{\rho,(f,g,1)}(A),\\
\mathrm{Corr}_{\rho,(f,g)}(A,B) &:= \mathrm{Corr}_{\rho,(f,g,1)}(A,B).
\end{aligned}$$

In [4, 5], the following bound played central roles in the uncertainty relations:

$$\min_{x,y\in[0,1]} L_{(f,g)}(x,y) \geq 4\beta_{(f,g)}, \tag{2.5}$$

where

$$L_{(f,g)}(x,y) = \frac{\big(f(x)^2-f(y)^2\big)\big(g(x)^2-g(y)^2\big)}{\big(f(x)g(x)-f(y)g(y)\big)^2}, \tag{2.6}$$

and

$$\beta_{(f,g)} := \min\left\{\frac{m}{(1+m)^2}, \frac{M}{(1+M)^2}\right\}, \tag{2.7}$$

and $m = \inf_{0<x<1}\frac{G'(x)}{F'(x)}$, $M = \sup_{0<x<1}\frac{G'(x)}{F'(x)}$ with $F(x) = \log f(x)$ and $G(x) = \log g(x)$. See the proof of Proposition 3.1 in [4].

In order to get Heisenberg-type uncertainty relation associated to the monotone triple skew information, which improves the result of [4], Yanagi and Kajihara considered the following assumption [10]:

Assumption 2.4. (f, g) and (f, h) are CLI monotone pairs satisfying

$$1+\frac{G(y)-G(x)}{F(y)-F(x)} \leq \frac{H(y)-H(x)}{F(y)-F(x)} \text{ for } x<y,$$

or equivalently, $1+\frac{G'(x)}{F'(x)} \leq \frac{H'(x)}{F'(x)}$, $x \in (0,1)$. Under the Assumption 2.4, Yanagi and Kajihara obtained the following bound:

$$\min_{x,y\in[0,1]} L_{(f,g,h)}(x,y) \geq 16\beta_{(f,g,h)}, \tag{2.8}$$

where

$$L_{(f,g,h)}(x,y) = \frac{(f(x)^2-f(y)^2)(g(x)^2-g(y)^2)(h(x)+h(y))^2}{(f(x)g(x)h(x)-f(y)g(y)h(y))^2} \tag{2.9}$$

and

$$\beta_{(f,g,h)} := \min\left\{\frac{m}{(1+m+n)^2}, \frac{m}{(1+m+N)^2}, \frac{M}{(1+M+n)^2}, \frac{M}{(1+M+N)^2}\right\}. \tag{2.10}$$

Here $n = \inf_{0<x<1} \frac{H'(x)}{F'(x)}$, $N = \sup_{0<x<1} \frac{H'(x)}{F'(x)}$, and m, M are defined in (2.7), $F(x) = \log f(x)$, $G(x) = \log g(x)$ and $H(x) = \log h(x)$. See the proof of Proposition 4.1 in [10].

Remark 2.5. In fact in [10] Yanagi and Kajihara showed that the inequality (2.8) also holds under the condition that (f, g) is a CLI monotone pair and (f, h) is a CLI anti-monotone pair satisfying

$$1 + \frac{G(y) - G(x)}{F(y) - F(x)} + \frac{H(y) - H(x)}{F(y) - F(x)} \geq 0 \quad \text{for } x < y, \tag{2.11}$$

or equivalently $1 + \frac{G'(x)}{F'(x)} + \frac{H'(x)}{F'(x)} \geq 0$, $x \in (0, 1)$. If $h = 1$ and (f, g) is a non-decreasing CLI monotone pair, then (2.11) holds and hence we get the inequality (2.8).

In [4], we have generalized the uncertainty relation on Wigner-Yanase-Dyson skew information as follows:

$$U_{\rho,(f,g)}(A)U_{\rho,(f,g)}(B) \geq \beta_{(f,g)}|\mathrm{Tr}\big(f(\rho)g(\rho)[A, B]\big)|^2, \ A, B \in M_{n,sa}. \tag{2.12}$$

Recall that the Wigner-Yanase-Dyson skew information and uncertainty relation corresponds to the case that $f(x) = x^{\alpha}$ and $g(x) = x^{1-\alpha}$ in (2.12).

In [10], using the monotone triple skew information $I_{\rho,(f,g,h)}(A)$, Yanagi and Kajihara have obtained the following uncertainty relation:

Theorem 2.6. *Under the Assumption 2.4, the following inequality holds.*

$$U_{\rho,(f,g,h)}(A)U_{\rho,(f,g,h)}(B) \geq \beta_{(f,g,h)}|\,Tr\big(f(\rho)g(\rho)h(\rho)[A, B]\big)|^2 \tag{2.13}$$

for $A, B \in M_{n,sa}$, *where* $\beta_{(f,g,h)}$ *is given in* (2.10).

Notice that when $h = 1$, (2.13) reduces to (2.12). See Remark 2.8.

On the other hand, in [5], we have obtained Schrödinger-type uncertainty relation for the monotone pair skew information as follows:

$$U_{\rho,(f,g)}(A)U_{\rho,(f,g)}(B) \geq 4\beta_{(f,g)}|\mathrm{Corr}_{\rho,(f,g)}(A, B)|^2, \ A, B \in M_{n,sa}. \tag{2.14}$$

In this paper we aim to extend Schrödinger-type uncertainty relation (2.14) for the monotone triple skew information $I_{\rho,(f,g,h)}(A)$. The following is our main result.

Theorem 2.7. *Let* (f, g, h) *be a triple of nonnegative operator monotone functions on* $[0, 1]$ *and suppose that*

$$f(x),\ g(x),\ \text{and } h(x) \text{ are non-decreasing on } (0, 1). \tag{2.15}$$

We assume that the triple (f, g, h) *satisfies Assumption 2.4 or* $h = 1$ *and* (f, g) *is a CLI monotone pair. Then for any* $\rho \in D_n^1$, *the following inequality holds.*

$$U_{\rho,(f,g,h)}(A)U_{\rho,(f,g,h)}(B) \geq 4\beta_{(f,g,h)}|\mathrm{Corr}_{\rho,(f,g,h)}(A, B)|^2 \tag{2.16}$$

for $A, B \in M_{n,sa}$.

The proof of this theorem is given in Section 3.

Remark 2.8. (a) When $h = 1$, since

$$L_{(f,g,1)}(x,y) = 4L_{(f,g)}(x,y) \text{ and } \beta_{(f,g,1)} = \beta_{(f,g)}, \tag{2.17}$$

we conclude that Theorem 2.7 extends the result of [5]. Particularly if $f(x) = x^{\alpha}$, $g(x) = x^{1-\alpha}$ $(0 < \alpha < 1)$, and $h = 1$, the triple skew information is the Wigner-Yanase-Dyson skew information.

(b) Let $f(x) = x^{\alpha}$, $g(x) = x^{\beta}$, $h(x) = x^{\gamma}$ with $0 < \alpha, \beta$ and $\alpha + \beta \le \gamma$ on $[0, 1]$. f, g, and h satisfy the conditions of Theorem 2.7 and $\beta_{(f,g,h)} = \frac{\alpha\beta}{(\alpha+\beta+\gamma)^2}$. See Corollary 4.2(1) of [10]. In particular, in case $\gamma = 1-\alpha-\beta$, the triple skew information is called the generalized Wigner-Yanase-Dyson skew information. For the generalized Wigner-Yanase-Dyson skew information, the uncertainty relations have been discussed in [3, 9], and the convexity was discussed in [1].

3. Proof of Theorem 2.7

In this section we give a proof of Theorem 2.7. We adopt a similar method used in [5] and [10].

Let $\rho = \sum_l \lambda_l |\phi_l\rangle\langle\phi_l| \in D_n^1$, $0 < \lambda_1 \le \cdots \le \lambda_n \le 1$, where $\{\phi_l\}_{l=1}^n$ is an orthonormal set in $\mathbb{C}^n$ and $\mathrm{Tr}(\rho) = \sum_l \lambda_l = 1$. Let f, g, and h be a triple of operator monotone functions satisfying the conditions of Theorem 2.7. By a simple calculation, we have for any $A, B \in M_{n,sa}$, denoting $a_{ml} = \langle\phi_m, A_0\phi_l\rangle$ and $b_{ml} = \langle\phi_m, B_0\phi_l\rangle$,

$$\begin{aligned}
&\mathrm{Tr}(f(\rho)g(\rho)h(\rho)A_0B_0) \\
&= \sum_l f(\lambda_l)g(\lambda_l)h(\lambda_l)\langle A_0\phi_l, B_0\phi_l\rangle \\
&= \sum_{l,m} f(\lambda_l)g(\lambda_l)h(\lambda_l)\langle A_0\phi_l, \phi_m\rangle\langle\phi_m, B_0\phi_l\rangle \\
&= \sum_{l<m} \Big(f(\lambda_l)g(\lambda_l)h(\lambda_l)a_{lm}b_{ml} + f(\lambda_m)g(\lambda_m)h(\lambda_m)a_{ml}b_{lm}\Big) \\
&\quad + \sum_l f(\lambda_l)g(\lambda_l)h(\lambda_l)a_{ll}b_{ll},
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\mathrm{Tr}(A_0f(\rho)g(\rho)B_0h(\rho)) \\
&= \sum_{l,m} f(\lambda_m)g(\lambda_m)h(\lambda_l)\langle A_0\phi_l, \phi_m\rangle\langle\phi_m, B_0\phi_l\rangle \\
&= \sum_{l<m} \Big(f(\lambda_m)g(\lambda_m)h(\lambda_l)a_{lm}b_{ml} + f(\lambda_l)g(\lambda_l)h(\lambda_m)a_{ml}b_{lm}\Big) \\
&\quad + \sum_l f(\lambda_l)g(\lambda_l)h(\lambda_l)a_{ll}b_{ll},
\end{aligned} \tag{3.2}$$

$$\mathrm{Tr}(A_0f(\rho)B_0g(\rho)h(\rho)) \tag{3.3}$$

$$= \sum_{l,m} f(\lambda_m)g(\lambda_l)h(\lambda_l)\langle A_0\phi_l, \phi_m\rangle\langle\phi_m, B_0\phi_l\rangle$$
$$= \sum_{l<m} \Big(f(\lambda_m)g(\lambda_l)h(\lambda_l)a_{lm}b_{ml} + f(\lambda_l)g(\lambda_m)h(\lambda_m)a_{ml}b_{lm}\Big)$$
$$+ \sum_{l} f(\lambda_l)g(\lambda_l)h(\lambda_l)a_{ll}b_{ll},$$

and

$$\mathrm{Tr}(A_0 g(\rho)B_0 f(\rho)h(\rho)) \tag{3.4}$$
$$= \sum_{l,m} f(\lambda_l)g(\lambda_m)h(\lambda_l)\langle A_0\phi_l, \phi_m\rangle\langle\phi_m, B_0\phi_l\rangle$$
$$= \sum_{l<m} \Big(f(\lambda_l)g(\lambda_m)h(\lambda_l)a_{lm}b_{ml} + f(\lambda_m)g(\lambda_l)h(\lambda_m)a_{ml}b_{lm}\Big)$$
$$+ \sum_{l} f(\lambda_l)g(\lambda_l)h(\lambda_l)a_{ll}b_{ll}.$$

From (2.1)-(2.2), (2.4), and (3.1)-(3.4) we get

$$I_{\rho,(f,g,h)}(A) = \frac{1}{2}\sum_{l<m}(f(\lambda_m) - f(\lambda_l))(g(\lambda_m) - g(\lambda_l))(h(\lambda_m) + h(\lambda_l))|a_{ml}|^2,$$
$$J_{\rho,(f,g,h)}(A) \geq \frac{1}{2}\sum_{l<m}(f(\lambda_m) + f(\lambda_l))(g(\lambda_m) + g(\lambda_l))(h(\lambda_m) + h(\lambda_l))|a_{ml}|^2,$$

and

$$\mathrm{Corr}_{\rho,(f,g,h)}(A,B) = \frac{1}{2}\sum_{l<m}(f(\lambda_m) - f(\lambda_l))(g(\lambda_m) - g(\lambda_l))h(\lambda_l)a_{lm}b_{ml}$$
$$+ \frac{1}{2}\sum_{l<m}(f(\lambda_m) - f(\lambda_l))(g(\lambda_m) - g(\lambda_l))h(\lambda_m)a_{ml}b_{lm}. \tag{3.5}$$

Proof of Theorem 2.7. Notice that $|a_{ml}| = |a_{lm}|$ and $|b_{ml}| = |b_{lm}|$. Since f and g are non-decreasing, we get from (3.5) that

$$|\mathrm{Corr}_{\rho,(f,g,h)}(A,B)|$$
$$\leq \frac{1}{2}\sum_{l<m}(f(\lambda_m) - f(\lambda_l))(g(\lambda_m) - g(\lambda_l))(h(\lambda_l) + h(\lambda_m))|a_{lm}||b_{ml}|$$

for any $A, B \in M_{n,sa}$. By condition (2.15), we have

$$|\mathrm{Corr}_{\rho,(f,g,h)}(A,B)|$$
$$\leq \frac{1}{4}\sum_{l<m}(f(\lambda_m) - f(\lambda_l))(g(\lambda_m) + g(\lambda_l))(h(\lambda_l) + h(\lambda_m))|a_{lm}||b_{ml}|$$
$$+ \frac{1}{4}\sum_{l<m}(f(\lambda_m) + f(\lambda_l))(g(\lambda_m) - g(\lambda_l))(h(\lambda_l) + h(\lambda_m))|a_{lm}||b_{ml}|$$

$$= \frac{1}{2}\sum_{l<m}(f(\lambda_m)g(\lambda_m) - f(\lambda_l)g(\lambda_l))(h(\lambda_l) + h(\lambda_m))|a_{lm}||b_{ml}|$$

$$\leq \sum_{l<m}(f(\lambda_m)g(\lambda_m)h(\lambda_m) - f(\lambda_l)g(\lambda_l)h(\lambda_l))|a_{lm}||b_{ml}|, \tag{3.6}$$

where we have used for $l < m$

$$0 \leq f(\lambda_m)g(\lambda_m)h(\lambda_l) \leq f(\lambda_m)g(\lambda_m)h(\lambda_m),$$
$$0 \leq f(\lambda_l)g(\lambda_l))h(\lambda_l) \leq f(\lambda_l)g(\lambda_l)h(\lambda_m).$$

Under the hypotheses of the theorem the inequality (2.8) holds. Thus

$$|\mathrm{Corr}_{\rho,(f,g,h)}(A,B)|$$
$$\leq \sum_{l<m} |f(\lambda_m)g(\lambda_m)h(\lambda_m) - f(\lambda_l)g(\lambda_l)h(\lambda_l)\,||a_{lm}||b_{lm}|$$
$$\leq \frac{1}{4\sqrt{\beta_{(f,g,h)}}}\sum_{l<m}\sqrt{\big(f(\lambda_m)^2 - f(\lambda_l)^2\big)\big(g(\lambda_m)^2 - g(\lambda_l)^2\big)(h(\lambda_m) + h(\lambda_l))^2}|a_{lm}b_{lm}|.$$

By Schwarz inequality, we have

$$16\beta_{(f,g,h)}|\mathrm{Corr}_{\rho,(f,g,h)}(A,B)|^2$$
$$\leq \sum_{l<m}(f(\lambda_m) - f(\lambda_l))(g(\lambda_m) - g(\lambda_l))(h(\lambda_m) + h(\lambda_l))|a_{ml}|^2$$
$$\times \sum_{l<m}(f(\lambda_m) + f(\lambda_l))(g(\lambda_m) + g(\lambda_l))(h(\lambda_m) + h(\lambda_l))|b_{lm}|^2$$
$$\leq 4I_{\rho,(f,g,h)}(A)J_{\rho,(f,g,h)}(B)$$

and similarly

$$16\beta_{(f,g,h)}|\mathrm{Corr}_{\rho,(f,g,h)}(A,B)|^2 \leq 4I_{\rho,(f,g,h)}(B)J_{\rho,(f,g,h)}(A).$$

From the above two inequalities we get

$$4\beta_{(f,g,h)}|\mathrm{Corr}_{\rho,(f,g,h)}(A,B)|^2 \leq U_{\rho,(f,g,h)}(A)U_{\rho,(f,g,h)}(B). \qquad \square$$

4. Concluding remarks

In this paper we have discussed Schrödinger-type uncertainty relation for the monotone triple skew information. It was motivated by the work of [10] where Heisenberg-type uncertainty relation for the same information has been studied. The monotone triple skew information, at first glance, looks as if it is a very general quantum information quantity. But, in order that a skew information to be physically meaningful, it should satisfy the so called convexity with respect to the states. It means that the information content decreases as two states mix. It seems that it is in general very hard to show the convexity for the general skew informations. As for the monotone triple skew information, we know only one example of the generalized Wigner-Yanase-Dyson skew information which was mentioned in Remark 2.8(b). It would be very nice if we knew other examples of monotone triple skew information which satisfy the convexity.

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Chul Ki Ko
University College
Yonsei University
Incheon 21983, Korea
E-mail address: kochulki@yonsei.ac.kr

Hyun Jae Yoo
Department of Applied Mathematics
Hankyong National University
Anseong 17579, Korea
E-mail address: yoohj@hknu.ac.kr